Moments of $\{x_n\}$

The purpose of this subsection is to see if something can be said about the ratio $(\sum_k k\,x_k)/(\sum_k x_k)$ when $\{x_k\}_{k\geq 1}$ is a non-zero solution to the system

$$0 = (1-q)\sum_{n\geq 1} \phi a_n x_n - (q\phi a_1 + d_1)x_1$$
$$0 = q\,\phi a_{n-1} x_{n-1} - (q\phi a_n + d_n)\,x_n .$$

(1.28)

with ϕ a suitable constant. To this end, introduce by way of notation $\varsigma = \sum_{n\geq 1}\phi a_n x_n$. The equations in (1.28) can be used to derive two expressions for x_n, these being

- $x_n = \frac{1}{(q\phi a_n + d_n)}\,(\prod_{j+1\leq k<n} \frac{q\phi a_k}{(q\phi a_k + d_k)})\,q\phi a_j\,x_j$ *for* $n \geq 2$.
- $x_n = \frac{1}{(q\phi a_n + d_n)}\,(\prod_{1\leq k<n} \frac{q\phi a_k}{(q\phi a_k + d_k)})(1-q)\,\varsigma$.

(1.29)

Note that ϕ must be such that

$$\frac{1-q}{q} \sum_{\text{[redacted]}} \prod_{k=1}^{n} \frac{q\phi a_k}{(q\phi a_k + d_k)} = 1 .$$

(1.30)

This last condition can be restated as saying that

$$\sum_{n\geq 2} (q\phi a_n + d_n)\,x_n = q\varsigma .$$

(1.31)

and therefore

$$q\varsigma - (q\phi\, a_1 + d_1)x_1 + \sum_{n\geq 1} d_n\, x_n = q\varsigma .$$

(1.32)

This tells us that

$$\sum_{n\geq 1} d_n x_n = (q\phi\, a_1 + d_1)\, x_1 = (1-q)\,\varsigma ,$$

(1.33)

where the left hand inequality comes via the n = 1 version of (1.29).

What is written in (1.33) is of at least two identities involving 'moments' of $\{x_n\}$. To elaborate, introduce a variable t and use (1.29) to see the equality between the following two formal series:

$$\sum_{n\geq 2} t^{n-1}\,((q\phi a_n + d_n)\,x_n) = \sum_{n\geq 1} t^n\,(q\phi a_n x_n) .$$

(1.35)

Let $\mathcal{Q}(t)$ denote the series $\sum_{n\geq 1} t^n\,(q\phi a_n x_n)$ and let $\wp(t)$ denote $\sum_{n\geq 1} t^n d_n x_n$. Then (1.35) says that

$$t^{-1}\mathcal{Q}(t) + t^{-1}\wp(t) = \mathcal{Q}(t) + (q\phi a_1 + d_1)\, x_1 \;. \tag{1.36}$$

This in turn can be rewritten using (1.33) as

$$\wp(t) = (t-1)\,\mathcal{Q}(t) + t\,(1-q)\,\varsigma \;. \tag{1.37}$$

Taking t = 1 on both sides recovers (1.33): $\sum_{n\geq1} d_n x_n = (1-q)\,\varsigma$. Differentiating once and setting t = 1 finds

$$\sum_{n\geq1} n\, d_n x_n = \mathcal{Q}(1) + (1-q)\,\varsigma. \tag{1.38}$$

To go further, use (1.31) to see that

$$\mathcal{Q}(1) = -\sum_{n\geq1} d_n x_n + q\,\varsigma + (q\phi a_1 + d_1)\, x_1 = q\varsigma \;, \tag{1.39}$$

Granted this last equality, then (1.38) asserts that

$$\sum_{n\geq1} n\, d_n x_n = \varsigma \;. \tag{1.40}$$

This with (1.33) says that

$$\frac{\sum_{n\geq1} n d_n x_n}{\sum_{n\geq1} d_n x_n} = \frac{1}{(1-q)} \tag{1.41}$$

In the case $d_n = d$ for all n, this asserts what is conjectured by Martin.

Identities for ‘moments’ of the form $\sum_{n\geq1} n^p d_n x_n$ for $p \geq 2$ require knowing something of the (p-1)’st derivative of $\mathcal{Q}$ at t = 1. I don’t know any good way to obtain these.