

Moments of $\{x_n\}$

The purpose of this subsection is to see if something can be said about the ratio $(\sum_k k x_k)/(\sum_k x_k)$ when $\{x_k\}_{k \geq 1}$ is a non-zero solution to the system

$$\begin{aligned} 0 &= (1-q) \sum_{n \geq 1} \phi a_n x_n - (q\phi a_1 + d_1) x_1 \\ 0 &= q\phi a_{n-1} x_{n-1} - (q\phi a_n + d_n) x_n. \end{aligned} \quad (1.28)$$

with ϕ a suitable constant. To this end, introduce by way of notation $\zeta = \sum_{n \geq 1} \phi a_n x_n$. The equations in (1.28) can be used to derive two expressions for x_n , these being

$$\begin{aligned} \bullet \quad x_n &= \frac{1}{(q\phi a_n + d_n)} \left(\prod_{j+1 \leq k < n} \frac{q\phi a_k}{(q\phi a_k + d_k)} \right) q\phi a_j x_j \text{ for } n \geq 2. \\ \bullet \quad x_n &= \frac{1}{(q\phi a_n + d_n)} \left(\prod_{1 \leq k < n} \frac{q\phi a_k}{(q\phi a_k + d_k)} \right) (1-q) \zeta. \end{aligned} \quad (1.29)$$

Note that ϕ must be such that

$$\frac{1-q}{q} \sum_{n \geq 1} \prod_{k=1}^n \frac{q\phi a_k}{(q\phi a_k + d_k)} = 1. \quad (1.30)$$

This last condition can be restated as saying that

$$\sum_{n \geq 2} (q\phi a_n + d_n) x_n = q\zeta. \quad (1.31)$$

and therefore

$$q\zeta - (q\phi a_1 + d_1) x_1 + \sum_{n \geq 1} d_n x_n = q\zeta. \quad (1.32)$$

This tells us that

$$\sum_{n \geq 1} d_n x_n = (q\phi a_1 + d_1) x_1 = (1-q) \zeta, \quad (1.33)$$

where the left hand inequality comes via the $n = 1$ version of (1.29).

What is written in (1.33) is of at least two identities involving ‘moments’ of $\{x_n\}$. To elaborate, introduce a variable t and use (1.29) to see the equality between the following two formal series:

$$\sum_{n \geq 2} t^{n-1} ((q\phi a_n + d_n) x_n) = \sum_{n \geq 1} t^n (q\phi a_n x_n). \quad (1.35)$$

Let $\mathcal{Q}(t)$ denote the series $\sum_{n \geq 1} t^n (q\phi a_n x_n)$ and let $\mathcal{P}(t)$ denote $\sum_{n \geq 1} t^n d_n x_n$. Then (1.35) says that

$$t^{-1} Q(t) + t^{-1} \wp(t) = Q(t) + (q\phi a_1 + d_1) x_1 . \quad (1.36)$$

This in turn can be rewritten using (1.33) as

$$\wp(t) = (t-1) Q(t) + t(1-q) \zeta . \quad (1.37)$$

Taking $t = 1$ on both sides recovers (1.33): $\sum_{n \geq 1} d_n x_n = (1-q) \zeta$. Differentiating once and setting $t = 1$ finds

$$\sum_{n \geq 1} n d_n x_n = Q(1) + (1-q) \zeta . \quad (1.38)$$

To go further, use (1.31) to see that

$$Q(1) = -\sum_{n \geq 1} d_n x_n + q \zeta + (q\phi a_1 + d_1) x_1 = q \zeta , \quad (1.39)$$

Granted this last equality, then (1.38) asserts that

$$\sum_{n \geq 1} n d_n x_n = \zeta . \quad (1.40)$$

This with (1.33) says that

$$\frac{\sum_{n \geq 1} n d_n x_n}{\sum_{n \geq 1} d_n x_n} = \frac{1}{(1-q)} \quad (1.41)$$

In the case $d_n = d$ for all n , this asserts what is conjectured by Martin.

Identities for ‘moments’ of the form $\sum_{n \geq 1} n^p d_n x_n$ for $p \geq 2$ require knowing something of the $(p-1)$ ’st derivative of Q at $t = 1$. I don’t know any good way to obtain these.