

Coming together

Martin considers two sorts of *coming together* models. I will start by analyzing Model 2 which is defined by the equations

$$\begin{aligned}
 & \bullet \quad x_1' = \sum_{n \geq 1} a_n x_n - x_1 (\beta \sum_{n \geq 1} x_n + \beta x_1 + \phi) \\
 & \bullet \quad x_n' = \beta x_1 x_{n-1} - (\beta x_1 + \phi) x_n. \\
 & \bullet \quad y' = (b - \phi) y. \\
 & \bullet \quad \phi = b y + \sum_n a_n x_n.
 \end{aligned}
 \tag{2.1}$$

The proposition that follows summarizes what can be said at this point. By way of notation, the proposition uses x_* to denote the unique, positive solution to the cubic equation $x(x+1)^2 = \beta/b$.

Proposition 1: *Suppose that $a_2 > 2b$ and $a_3 - a_2 > b$. Then there is at most one equilibrium solution to (2.1) with both $y > 0$ and $x_1 > 0$. If, in addition, either*

$$a_1 \geq b \quad \text{or} \quad (x_* + 1)^2 < \sum_{n \geq 0} \hat{a}_{n+1} \left(\frac{x_*}{x_* + 1} \right)^n,$$

then there are no $y > 0$ and $x_1 > 0$ equilibrium solutions. On the other hand, if the three conditions $a_2 > 2b$ and $a_3 - a_2 > b$ and $a_1 < b$ hold, then there exists one and only one equilibrium solution to (2.1) with both $y > 0$ and $x_1 > 0$ solution.

As noted by Martin, this definition of ϕ has $y + \sum_{n \geq 1} n x_n = 1$ at an equilibrium point. It is also the case that a solution with $x_1 > 0$ has all $x_{n \geq 1}$ positive also.

Proof of Proposition 1: An equilibrium point with both $y \neq 0$ and $x_1 \neq 0$ is characterized by the conditions:

$$\begin{aligned}
 & \bullet \quad x_1 = \phi \zeta / (\beta x_1 + \phi)^2 \\
 & \bullet \quad x_n = \alpha x_{n-1} = \alpha^{n-1} x_1 \text{ where } \alpha = \beta x_1 / (\beta x_1 + \phi). \\
 & \bullet \quad \phi = b.
 \end{aligned}
 \tag{2.2}$$

Here, ζ is shorthand for $\sum_{n \geq 1} a_n x_n$. Note in this regard that the condition on x_1 can be derived from the top bullet in (2.1) by noting that

$$\sum_{n \geq 1} x_n = \frac{1}{1-\alpha} x_1 = \phi^{-1} (\beta x_1 + \phi) x_1
 \tag{2.3}$$

To see what to make of the equilibria, use the third bullet in (2.2) to set $\phi = b$ in the second bullet and so write $\alpha = \beta x_1 / (\beta x_1 + b)$ and thus x_n as a function of x_1 . This done, then the top bullet in (2.2) defines x_1 and y as solutions to the equations

$$\begin{aligned} \bullet \quad y &= 1 - b^{-1} x_1 \sum_{n \geq 1} a_n \left(\frac{x_1}{x_1 + \eta} \right)^{n-1} \\ \bullet \quad y &= 1 - \eta^{-2} x_1 (x_1 + \eta)^2. \end{aligned} \tag{2.4}$$

where $\eta = b/\beta$. The equations in (2.4) imply that an equilibrium point obeys

$$(x_1 + \eta)^2 = \eta^2 b^{-1} \sum_{n \geq 0} a_{n+1} \left(\frac{x_1}{x_1 + \eta} \right)^n. \tag{2.5}$$

By renaming $x_1 = \eta x$ and $a_n = b \hat{a}_n$, the equation in (2.5) can be written as

$$0 = -(x+1)^2 + \sum_{n \geq 0} \hat{a}_{n+1} \left(\frac{x}{x+1} \right)^n. \tag{2.6}$$

with the constraint that $x(x+1)^2 \leq \eta^{-1}$ to have $y \neq 0$.

The expression in (2.6) can be written as

$$0 = \hat{a}_1 - 1 + (\hat{a}_2 - 2)x + (\hat{a}_3 - \hat{a}_2 - 1)x^2 + h(x) \tag{2.7}$$

where $h(x) > 0$ and $h'(x) > 0$ for $x \geq 0$. Moreover, $h(x) \sim x^3$ for x near 0. Note for reference momentarily that the derivative of the right hand side is

$$\hat{a}_2 - 2 + 2(\hat{a}_3 - \hat{a}_2 - 1)x + h'(x). \tag{2.8}$$

This last expression is positive if both $\hat{a}_2 - 2$ and $\hat{a}_3 - \hat{a}_2 - 1$ are positive. It follows as a consequence that if this condition holds, then there is at most 1 solution to (2.7).

Meanwhile, the right hand side of (2.7) at $x = 0$ is $\hat{a}_1 - 1$. These last two observations lead to the following:

- *There are no $x > 0$ solutions to (2.6) if: $\hat{a}_1 \geq 1$ and $\hat{a}_2 \geq 2$ and $\hat{a}_3 - \hat{a}_2 - 1 \geq 0$.*
 - *There exists a unique $x > 0$ solution to (2.6) if: $\hat{a}_1 < 1$ and $\hat{a}_2 \geq 2$ and $\hat{a}_3 - \hat{a}_2 - 1 > 0$.*
- (2.9)

It remains to be seen when the solution to (2.6) is such that $x(x+1)^2 < \eta^{-1}$. To this end, let x_* denote the one real, positive solution to the cubic equation $x(x+1)^2 - \eta^{-1} = 0$. As noted above, if both $\hat{a}_2 \geq 2$ and $\hat{a}_3 - \hat{a}_2 - 1 \geq 0$, then the right hand side of (2.10) is an increasing function of x . This being the case, there exists a $y > 0$ and $x_1 > 0$ equilibrium point if both the second bullet in (2.9) holds and if

$$(x_0 + 1)^2 < \sum_{n \geq 0} \hat{a}_{n+1} \left(\frac{x_0}{x_0 + 1} \right)^n. \tag{2.10}$$

This solution is unique. On the other hand, if $\hat{a}_2 \geq 2$ and $\hat{a}_3 - \hat{a}_2 - 1 \geq 0$ and if the inequality in (2.10) is reversed, then there are no $y > 0$ and $x_1 > 0$ solutions to (2.10).